Q No → Define closed and Continuous linear transformation. Give an example of closed transformation which is not Continuous linear transformation, R^n space is reflexive.

Ans. (Defn) Closed Transformation:- A linear transformations T which maps a subspace D of a Banach Space B into a Banach Space B′ is called closed if its graph $G(T) = \{(x, Tx) : x \in D\}$ is closed in $B \times B'$.

Hence, T is closed iff $x_n \to x$ & $Tx_n \to y$ imply $x \in D$, $y = Tx$. In this language the closed graph theorem may be stated as follows If B & B′ are Banach spaces and if T is a linear transformation of B into B′ then T is Continuous iff T is closed.

Continuous linear transformation:- Let E and F be normed linear spaces. A linear transformation T from E into F is said to be Continuous if T is Continuous as a map from the metric space E to the metric space F.

i.e. if for every $x \in E$ and every sequence $\{x_n\}$ in E Converging to x the sequence $\{T(x_n)\}$ Converges to $T(x)$, i.e. $\|x_n - x\| \to 0$ as $\|T(x_n) - T(x)\| \to 0$ as $n \to \infty$.

Example of a closed operator (transformation) which is not continuous

Let $X = C[0,1]$ and let $X = Y$. Let D be the set of $x \in X$ such that the derivative $x'(t)$ exists and is continuous on $[0,1]$. Then D is a subspace of X. Let A be an transformation (operator) with ~~domain~~ domain D defined by $Ax = x'$ where dash denotes the derivative. Then clearly A is a linear operator.

If $x_n(t) = t^n$ then $x_n'(t) = nt^{n-1}$ and $\|x_n\| = 1$ and $\|Ax_n\| = \|x_n'\| = n$. So A is not bounded. Consequently A is not continuous. But we now show that A is closed. We suppose that

$x_n \in D$, $x_n \to x$ and $Ax_n \to y$.

Then, $x_n(t)$ converges uniformly to $x(t)$ and $Ax_n = x_n'(t)$ converges uniformly to $y(t)$. Since $x_n'(t)$ is continuous, $y(t)$ is continuous on $[0,1]$. By a standard theorem on the theory of converges, it follows that $x(t)$ is differentiable and $x'(t) = y(t)$.

So, $x \in D$, and $Ax = x' = y$. This shows that A is closed.

R^n Space is Reflexive:-

Let $e_1 = (1,0,0,\ldots,0)$, $e_2 = (0,1,0,\ldots,0)$ ~~any element~~ $e_n = (0,0,0,\ldots,1)$ be a basis of R^n. Then any element $x = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in R^n$ can be written as,

$$x = \sum_{k=1}^{n} \alpha_k e_k.$$

If f is a continuous linear functional on $\mathbb{R}^n$, then $f(x) = f\left(\sum_{k=1}^{n} \alpha_k e_k\right) = \sum_{k=1}^{n} \alpha_k f(e_k) = \sum_{k=1}^{n} \alpha_k \beta_k$

where $\beta_k = f(e_k)$.

Conversely, every n-tuple $(\beta_1, --- \beta_n) \in \mathbb{R}^n$ determines a continuous linear function f of $\mathbb{R}^n$, given by $\underbrace{f(x) = \sum_{k=1}^{n} \alpha_k \beta_k}$, where $x = (\alpha_1, -- \alpha_n)$

By Cauchy-Schwarz inequality, we have

$$|f(x)| \leq \sum_{k=1}^{n} |\alpha_k| |\beta_k| \leq \left(\sum_{k=1}^{n} \alpha_k^2\right)^{1/2} \left(\sum_{k=1}^{n} \beta_k^2\right)^{1/2}$$

$$= \|x\| \left(\sum_{k=1}^{n} \beta_k^2\right)^{1/2}.$$

Hence, f is continuous and

$$\|f\| = \sup_{x \neq 0} \frac{|f(x)|}{\|x\|} \leq \left(\sum_{k=1}^{n} \beta_k^2\right)^{1/2}.$$

However, if $x = (\beta_1, --- \beta_n)$, then

$$f(x) = \sum_{k=1}^{n} \beta_k^2 \text{ and } \frac{|f(x)|}{\|x\|} = \left(\sum_{k=1}^{n} \beta_k^2\right)^{1/2}.$$

$$\therefore \|f\| = \left(\sum_{k=1}^{n} \beta_k^2\right)^{1/2} = \|y\|$$

where, $y = (\beta_1, \beta_2, -- \beta_n) \in \mathbb{R}^n$.

Hence the mapping from the dual space $(\mathbb{R}^n)^*$ onto $\mathbb{R}^n$ defined by

$$f \to y = (\beta_1, --- \beta_n).$$

is norm preserving, clearly the mapping

is linear, one-one and onto. Therefore, it is an isomorphism. Hence the conjugate space of R^n is R^n i.e. $(R^n)^* \doteq R^n$.

Thus, R^n is reflexive.

Ex. Q.No> Prove that every Hilbert Space is reflexive:-

Soln We have two natural mappings of H into H^{**}, the Second of which is an onto mapping. The Banach Space Canonical imbedding

$x \to F_x$ Where $F_x(f) = f(x)$

and the Product mapping

$x \to f_x \to F_{f_x}$, Where $f_x(y) = (y, x)$ and

$F_{f_x}(f) = (f, f_x)$.

The above two mappings are equal Since

$F_{f_x}(f) = (f, f_x) = (f_z, f_x)$ (let $f = f_z$)

$= (x, z) = f_z(x) = f(x) = F_x(f)$ for all $f \in H^*$

Hence, $F_{f_x} = F_x$.

Since the product mapping is a mapping of H onto H^{**} and since this mapping coincides with the canonical embedding of H to H^{**}, it follows that there is an isomorphism between H and H^{**}. Hence H is reflexive.